A GENERALIZATION OF THE LINE TRANSLATION THEOREM

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ABSTRACT. Through the method of brick decomposition and the operations on essential topological lines, we generalize the line translation theorem of Beguin, Crovisier, Le Roux [BCL] in the case where the property of preserving a finite measure with total support is replaced by the intersection property.

1. INTRODUCTION

Let $\mathbb{A} = \mathbb{R}/\mathbb{Z} \times \mathbb{R}$ be the open annulus. We denote π the covering map

$$\begin{array}{rcl} \pi : \mathbb{R}^2 & \to & \mathbb{A} \\ (x,y) & \mapsto & (x+\mathbb{Z}, y), \end{array}$$

and T the generator of the covering transformation group

$$\begin{array}{rcl} T : \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+1, y). \end{array}$$

Write respectively S and N for the lower and the upper end of $\mathbb{A}$. We call *essential line* in $\mathbb{A}$ every simple path, parametrized by $\mathbb{R}$, properly embedded in $\mathbb{A}$, joining one end to the other one. We call *essential circle* in $\mathbb{A}$ every simple closed curve which is not null-homotopic.

Let f be a homeomorphism of $\mathbb{A}$. We say that f satisfies the *intersection property* if any essential circle in $\mathbb{A}$ meets its image by f. We denote the space of all homeomorphisms of $\mathbb{A}$ which are isotopic to the identity as $\mathrm{Homeo}_*(\mathbb{A})$ and its subspace whose elements additionally have the intersection property as $\mathrm{Homeo}_*^{\wedge}(\mathbb{A})$. If X is a topological space and A is a subset of X, denote respectively by $\mathrm{Int}(A)$, $\mathrm{Cl}(A)$ and ∂A the interior, the closure and the boundary of A.

The goal of the paper is to generalize the line translation theorem of Béguin, Crovisier, Le Roux [BCL] in the case where the property of preserving a finite measure with total support is replaced by the intersection property. A similar result has done when $\mathbb{A}$ is a closed annulus due to Béguin, Crovisier, Le Roux and Patou [BCLP]. The reason why we consider the intersection property is that some interesting questions can be reduced to homeomorphisms of the open annulus which satisfy the intersection property but do not preserve a finite measure. For example, consider a homeomorphism f of a closed surface

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M with genus at least one that preserves a finite measure with total support. Take a lift F to the universal covering space $\widetilde{M}$ (homeomorphic to the Poincaré disk) of M and suppose that F has a fixed point. When we remove this point, we obtain a map of the open annulus that satisfies the intersection property but does not preserve a finite measure. We will strongly use the arguments of Beguin, Crovisier, Le Roux [BCL] but will have to add some crucial lemmas to weaken their theorem. Let us first recall some results and then state our main results.

When $f \in \mathrm{Homeo}_*(\mathbb{A})$, we define the rotation number of a positively recurrent point as follows (see [Lec1] for details). We say that a positively recurrent point z has a *rotation number* $\rho(F;z) \in \mathbb{R}$ for a lift F of f to the universal covering space $\mathbb{R}^2$ of $\mathbb{A}$, if for every subsequence $\{f^{n_k}(z)\}_{k\geq 0}$ of $\{f^n(z)\}_{n\geq 0}$ which converges to z, we have

$$\lim_{k\to+\infty} \frac{p_1 \circ F^{n_k}(\widetilde{z}) - p_1(\widetilde{z})}{n_k} = \rho(F;z)$$

where $\widetilde{z} \in \pi^{-1}(z)$ and p_1 is the first projection $p_1(x,y) = x$. In particular, the rotation number $\rho(F;z)$ always exists and is rational when z is a fixed or periodic point of f. Let $\mathrm{Rec}^+(f)$ be the set of positively recurrent points of f. We denote the set of rotation numbers of positively recurrent points of f as $\mathrm{Rot}(F)$.

It is well known that a positively recurrent point of f is also a positively recurrent point of f^q for all $q \in \mathbb{N}$ (we give a proof in the Appendix, see Lemma 18). By the definition of rotation number, we easily get that $\mathrm{Rot}(F)$ satisfies the following elementary properties.

1. $\mathrm{Rot}(T^k \circ F) = \mathrm{Rot}(F) + k$ for every $k \in \mathbb{Z}$;
2. $\mathrm{Rot}(F^q) = q\mathrm{Rot}(F)$ for every $q \in \mathbb{N}$.

We recall that a *Farey interval* is an interval of the form $]\frac{p}{q}, \frac{p'}{q'}[$ with $q, q' \in \mathbb{N} \setminus \{0\}$, $p, p' \in \mathbb{Z}$ and $qp' - pq' = 1$. Our main result is the following:

Theorem 1 (Generalization of the line translation theorem)**.** *Let $f \in \mathrm{Homeo}_*^{\wedge}(\mathbb{A})$ and F be a lift of f to $\mathbb{R}^2$. Assume that $\mathrm{Rot}(F) \neq \emptyset$ and its closure is contained in a Farey interval $]\frac{p}{q}, \frac{p'}{q'}[$. Then, there exists an essential line γ in $\mathbb{A}$ such that the lines $\gamma, f(\gamma), \cdots, f^{q+q'-1}(\gamma)$ are pairwise disjoint. Moreover, the cyclic order of these lines is the same as the cyclic order of the $q+q'-1$ first iterates of a vertical line $\{\theta\} \times \mathbb{R}$ under the rigid rotation with angle ρ, for any $\rho \in]\frac{p}{q}, \frac{p'}{q'}[$.*

It is easy to see that a homeomorphism of $\mathbb{A}$ that preserves a finite measure with total support satisfies the intersection property. Note that the statement of Theorem 1 with this stronger condition is exactly what is done in [BCL].

Let us consider the simple case of Theorem 1 when $(p,q) = (0,1)$ and $(p',q') = (1,1)$. We obtain the following corollary.

Corollary 2. *Let $f \in \mathrm{Homeo}_*^{\wedge}(\mathbb{A})$. We suppose that F is a lift of f to $\mathbb{R}^2$ and that*

$$\emptyset \neq \mathrm{Cl}(\mathrm{Rot}(F)) \subset]0,1[.$$

Then there exists an essential line in $\mathbb{A}$ that is free for f, that means disjoint from its image by f.

A similar result in the case when $\mathbb{A}$ is a closed annulus is known due to Bonatti and Guillou [G2]: if f is a homeomorphism of the closed annulus $\mathbb{R}/\mathbb{Z} \times [0,1]$ isotopic to the identity and fixed point free, then either there is a free simple path for f that joins the two boundary of the closed annulus or there is a free essential circle in the closed annulus for f.

As we will recall later, the rotation number set of F is a closed interval if f satisfies the intersection property and F is any lift of f to the universal covering space $\mathbb{R} \times [0,1]$. Therefore, if the map f has no fixed point, we can find a lift F of f to $\mathbb{R}/\mathbb{Z} \times [0,1]$ such that the rotation number set of F is contained in $]0,1[$.

As an immediate corollary of Theorem 1, we get the following generalization of the Corollary 0.3 in [BCL]:

Corollary 3. *We suppose that $f \in \mathrm{Homeo}_*^{\wedge}(\mathbb{A})$ has a rotation set reduced to a single irrational number ρ (for any given lift F of f). Then, for every $n \in \mathbb{N} \setminus \{0\}$, there exists an essential line γ in $\mathbb{A}$, such that the lines $\gamma, f(\gamma), \cdots, f^n(\gamma)$ are pairwise disjoint. The cyclic order of the lines $\gamma, f(\gamma), \cdots, f^n(\gamma)$ is the same as the cyclic order of the n first iterates of a vertical line under the rigid rotation of angle ρ.*

Theorem 1 above is a global result on the open annulus. A local version was studied by Patou in her thesis [P] under similar hypotheses of the line translation theorem in [BCL]. More precisely, she considered a local homeomorphism F between two neighborhoods of the origin O in $\mathbb{R}^2$ that fixes O. If F preserves the orientation and the area of $\mathbb{R}^2$, and it has no other periodic point except O in its domain, then for every $N \in \mathbb{N}$, there exists a simple arc Γ in a small neighborhood of O that issues from O such that the arcs $\Gamma, F(\Gamma), \cdots, F^N(\Gamma)$ are pairwise disjoint except at O.

We will introduce some mathematical objects and recall some well-known facts in Section 2. In Section 3, we first state some crucial lemmas without proof and then prove Theorem 1. In Section 4, we prove the lemmas stated in Section 3, and we give some remarks about the relations between the positively recurrent set and the rotation number set. In Section 5, we define a weak rotation number which is a generalization of the rotation number we have defined above. We prove the generalization of the line translation theorem in the weak sense. Finally, in Section 6, we provide a proof of a well known fact that we require in this paper but were unable to find in the literature.

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2. Preliminaries

2.1. **Essential topological lines in $\mathbb{R}^2$.** A *topological line* in $\mathbb{R}^2$ is the image of a proper continuous embedding of $\mathbb{R}$. Equivalently, using Schoenflies theorem, it is the image of an Euclidean line under a homeomorphism of $\mathbb{R}^2$. Let Γ be a topological line whose orientation is induced by a parametrization. We denote by $L(\Gamma)$ the connected component of $\mathbb{R}^2 \setminus \Gamma$ on the left of Γ, and by $R(\Gamma)$ the connected component of $\mathbb{R}^2 \setminus \Gamma$ on the right of Γ. We get a partial order relation $\leq$ on the set of oriented lines by writing: $\Gamma_1 \leq \Gamma_2$ if $L(\Gamma_1) \subset L(\Gamma_2)$

(or equivalently $R(\Gamma_2) \subset R(\Gamma_1)$). We get also a transitive relation $<$ by writing $\Gamma_1 < \Gamma_2$ if $\mathrm{Cl}(L(\Gamma_1)) \subset L(\Gamma_2)$ (or equivalently $\mathrm{Cl}(R(\Gamma_2)) \subset R(\Gamma_1)$).

We call *essential line* in $\mathbb{R}^2$ an oriented line Γ such that $\lim_{t\to+\infty} p_2(\Gamma(t)) = +\infty$ and $\lim_{t\to-\infty} p_2(\Gamma(t)) = -\infty$ where $p_2(x, y) = y$ and $t \mapsto \Gamma(t)$ is a parametrization. If Γ is a line and F is a homeomorphism of $\mathbb{R}^2$, then $F(\Gamma)$ is a line. Moreover, if Γ is oriented, then there is a natural orientation for $F(\Gamma)$. We say that an orientation preserving homeomorphism F of $\mathbb{R}^2$ is *essential* if for every essential line Γ in $\mathbb{R}^2$, $F(\Gamma)$ is also an essential line in $\mathbb{R}^2$. Any homeomorphism F that lifts a homeomorphism f of an open annulus $\mathbb{A}$ isotopic to the identity is an essential homeomorphism.

Let Γ_1 and Γ_2 be two essential lines in $\mathbb{R}^2$, and let U be the unique connected component of the set $L(\Gamma_1)\cap L(\Gamma_2)$ which contains half lines of the form $]-\infty, a[\times\{b\}$ for some numbers a and b. Then the boundary of U is an essential line in $\mathbb{R}^2$, denoted by $\Gamma_1 \vee \Gamma_2$ (the proof of this fact uses a classical result by B. Kerékjártó [K]. See [BCLP] and [BCL]).

Remark 1. Let $\Gamma_1, \Gamma_2, \Gamma_3$ be three essential lines in $\mathbb{R}^2$. The following properties are immediate consequences of the definition of the line $\Gamma_1 \vee \Gamma_2$.

(1) The line $\Gamma_1 \vee \Gamma_2$ is included in the union of the lines Γ_1 and Γ_2. Hence, if $\Gamma_3 < \Gamma_1$ and $\Gamma_3 < \Gamma_2$, then $\Gamma_3 < \Gamma_1 \vee \Gamma_2$.
(2) The sets $R(\Gamma_1)$ and $R(\Gamma_2)$ are included in the set $R(\Gamma_1 \vee \Gamma_2)$. In other words, we have $\Gamma_1 \vee \Gamma_2 \leq \Gamma_1$ and $\Gamma_1 \vee \Gamma_2 \leq \Gamma_2$.

2.2. **Brouwer theory.** A *Brouwer homeomorphism* is an orientation preserving fixed point free homeomorphism of $\mathbb{R}^2$. Given a Brouwer homeomorphism F, a *Brouwer line* for F is a topological line Γ, disjoint from $F(\Gamma)$ and separating $F(\Gamma)$ from $F^{-1}(\Gamma)$. The Brouwer Plane Translation Theorem asserts that every point belongs to a Brouwer line.

Now we introduce the following "equivariant" version of the Brouwer Plane Translation Theorem which has been proved by Guillou and Sauzet:

Theorem 4 ([G1],[S]). *Let $f \in \mathrm{Homeo}_*(\mathbb{A})$ and F be a lift of f to $\mathbb{R}^2$. If F is fixed point free, then there exists an essential circle in $\mathbb{A}$ that is free under f (and therefore that lifts to a Brouwer line of F) or there exists an essential line in $\mathbb{A}$ that lifts to a Brouwer line of F.*

Let Γ be a Brouwer line. It can be oriented such that $\Gamma < F(\Gamma)$. Since F preserves the orientation, we have $F^k(\Gamma) < F^{k+1}(\Gamma)$. By induction, we see that $F^p(\Gamma) < F^q(\Gamma)$ if and only if $p < q$. In particular, the lines $(F^k(\Gamma))_{k\in\mathbb{Z}}$ are pairwise disjoint.

Now let U be the open region of $\mathbb{R}^2$ situated between the lines Γ and $F(\Gamma)$, and $\mathrm{Cl}(U) = \Gamma \cup U \cup F(\Gamma)$. The sets $(F^k(U))_{k\in\mathbb{Z}}$ are pairwise disjoint. As a consequence, the restriction of F to the open set $O_U = \bigcup_{k\in\mathbb{Z}} F^k(\mathrm{Cl}(U))$ is conjugate to a translation. In particular, if the iterates of $\mathrm{Cl}(U)$ cover the whole plane, then F itself is conjugate to a translation.

2.3. **Franks' Lemma and brick decompositions.** A *free disk chain* for a homeomorphism F of $\mathbb{R}^2$ is a finite set b_i $(i = 1, 2, \cdots, n)$ of embedded open disks in $\mathbb{R}^2$ satisfying

(1) $F(b_i) \cap b_i = \emptyset$ for $1 \leq i \leq n$;
(2) if $i \neq j$ then either $b_i = b_j$ or $b_i \cap b_j = \emptyset$;

(3) for $1 \leq i \leq n$, there exists $m_i > 0$ such that $F^{m_i}(b_i) \cap b_{i+1} \neq \emptyset$.

We say that $(b_i)_{i=1}^n$ is a *periodic free disk chain* if $b_1 = b_n$.

In [F1], Franks got the following useful lemma about the existence of fixed points of an orientation preserving homeomorphism F of $\mathbb{R}^2$ from Brouwer theory.

Proposition 5 (Franks' Lemma)**.** *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation preserving homeomorphism which possesses a periodic free disk chain. Then F has at least one fixed point.*

A *brick decomposition* of a surface S (not necessarily closed) is given by a one dimensional stratified set Σ (the *skeleton* of the decomposition) with a zero dimensional submanifold V such that any vertex $v \in V$ is local the extremity of exactly three edges. A brick is the closure of a connected component of $S \setminus \Sigma$. Write $\mathfrak{B}$ for the set of bricks. For any $\mathfrak{X} \subset \mathfrak{B}$ the union of bricks which are in $\mathfrak{X}$ is a sub-surface of S with boundary. Suppose that F is a homeomorphism of S. We can define the relation $B\mathscr{R}B' \Leftrightarrow F(B) \cap B' \neq \emptyset$ where $B, B' \in \mathfrak{B}$, and write $B \preceq B'$ if there exists a sequence $(B_i)_{0 \leq i \leq n} \subset \mathfrak{B}$ such that $B_0 = B$, $B_n = B'$ and $B_i \mathscr{R} B_{i+1}$ for every $i \in \{0, \dots, n-1\}$. The union $B_{\succeq} = \bigcup_{B' \succeq B} B'$ (resp. $B_{\preceq} = \bigcup_{B' \preceq B} B'$) is a closed subset satisfying $F(B_{\succeq}) \subset \mathrm{Int}(B_{\succeq})$ (resp. $F^{-1}(B_{\preceq}) \subset \mathrm{Int}(B_{\preceq})$).

Suppose that $S = \mathbb{R}^2$ and F is an orientation preserving homeomorphism of $\mathbb{R}^2$ without fixed point, we say that $\mathfrak{B}$ is *free* if every brick $B \in \mathfrak{B}$ is free. The stronger version of Franks' Lemma given in Guillou and Le Roux [Ler1] asserts that there is no closed chain of bricks of $\mathfrak{B}$ if $\mathfrak{B}$ is free. This implies that $\preceq$ is a partial order. Furthermore, we can construct a *maximal free decomposition*: it is a brick decomposition with free bricks such that the union of two adjacent bricks is no more free [S]. The decomposition being maximal, two adjacent bricks are comparable. In fact, it appears that for every brick B, the union of bricks $B' \succeq B$ adjacent to B is non-empty, as is the union of adjacent bricks $B' \preceq B$. This implies that $B_{\succeq}$ and $B_{\preceq}$ are connected surfaces with boundary. The fact that we are working with bricks implies that $\partial(B_{\succeq})$ and $\partial(B_{\preceq})$ is a one dimensional manifold; the inclusion $F(B_{\succeq}) \subset \mathrm{Int}(B_{\succeq})$ implies that every component of $\partial(B_{\succeq})$ and $\partial(B_{\preceq})$ is a Brouwer line. If F commutes with T, we can further assume that the free decomposition $\mathfrak{B}$ is invariant by T (see [Lec2] for details). The relation $\preceq$ is T-equivalent: $B \preceq B' \Leftrightarrow T(B) \preceq T(B')$. However a maximal T-invariant free decomposition is not necessarily maximal among the free decomposition. For example, there is no T-invariant free decomposition for T^k ($k \geq 3$) which is maximal among the free decompositions (see [S] for details). Thus in this case, we can not assert that two adjacent bricks are comparable and that $B_{\succeq}$ and $B_{\preceq}$ are connected. But if there exists $p \in \mathbb{Z} \setminus \{0\}$ such that $T^p(B) \in B_{\succeq}$ (resp. $T^p(B) \in B_{\preceq}$), then the set $\bigcup_{k \in \mathbb{Z}} T^k(B_{\succeq})$ (resp. $\bigcup_{k \in \mathbb{Z}} T^k(B_{\preceq})$) is closed connected surface with boundary, which is a direct consequence of Proposition 2.7 in [Lec2]. Moreover, the image par F (resp. F^{-1}) of $\bigcup_{k \in \mathbb{Z}} T^k(B_{\succeq})$ (resp. $\bigcup_{k \in \mathbb{Z}} T^k(B_{\preceq})$) is contained in its interior.

3. Proof of the generalization of the line translation theorem

In this section, we first state some crucial lemmas and then prove the main result Theorem 1. We delay the proofs of these lemmas in the next section.

Lemma 6. *Let $H_1, H_2, ..., H_p$ be pairwise commuting essential homeomorphisms of $\mathbb{R}^2$ and $(q_i)_{1\leq i\leq q}$ be a family of positive integers. If there is an essential line Γ in $\mathbb{R}^2$ such that $\Gamma < H_i^{q_i}(\Gamma)$ for every $i \in \{1, \cdots, p\}$, then there is an essential line Γ' in $\mathbb{R}^2$ such that $\Gamma' < H_i(\Gamma')$ for every $i \in \{1, \cdots, p\}$.*

Lemma 7. *If $f \in \mathrm{Homeo}_*^\wedge(\mathbb{A})$, then $f^q \in \mathrm{Homeo}_*^\wedge(\mathbb{A})$ for any integer $q > 1$.*

Lemmas 6 and 7 are similar to Proposition 3.1 and Lemma 6.2 in [BCLP], respectively. Whilst, the following lemma is a key and new result given in this paper, by which we can directly prove Corollary 2 and thus we can furthermore prove Theorem 1.

Lemma 8. *Let $f \in \mathrm{Homeo}_*^\wedge(\mathbb{A})$. We suppose that F is a lift of f to $\mathbb{R}^2$ and that*

$$\emptyset \neq \mathrm{Rot}(F) \subset [-q+2, q-2],$$

where $q \geq 2$, then there exists an oriented essential line γ in $\mathbb{A}$ joining S to N that is lifted to an essential line Γ in $\mathbb{R}^2$ which satisfies

$$T^{-q}(\Gamma) < F(\Gamma) < T^q(\Gamma).$$

Proof of the Theorem 1. Firstly, we prove the special case where $(p, q) = (0, 1)$ and $(p', q') = (1, 1)$, that is Corollary 2. By the hypothesis, we can suppose that $\mathrm{Rot}(F) \subset [\frac{1}{n}, 1 - \frac{1}{n}]$ for some $n \in \mathbb{N}$ large enough. Hence the closure of $\mathrm{Rot}(F^{2n} \circ T^{-n})$ is contained in $[-n+2, n-2]$. The map f satisfies the intersection property, thus does f^{2n} by Lemma 7. By Lemma 8, there exists an oriented essential line γ in $\mathbb{A}$, joining S to N, that is lifted to an essential line Γ in $\mathbb{R}^2$ which satisfies $T^{-n}(\Gamma) < (F^{2n} \circ T^{-n})(\Gamma) < T^n(\Gamma)$. This implies that $\Gamma < F^{2n}(\Gamma) < T^{2n}(\Gamma)$.

By Lemma 6, let $H_1 = F$, $H_2 = F^{-1} \circ T$ and $q_1 = q_2 = 2n$, we can construct an essential line Γ' in $\mathbb{R}^2$ which satisfies $\Gamma' < F(\Gamma')$ and $\Gamma' < (F^{-1} \circ T)(\Gamma')$, so $\Gamma' < F(\Gamma') < T(\Gamma')$. We deduce that $\gamma' = \pi(\Gamma')$ is an essential line in $\mathbb{A}$ satisfying the conclusion of Corollary 2.

Secondly, we turn to prove the general case. Let $\Phi = T^{-p} \circ F^q$ and $\Psi = T^{p'} \circ F^{-q'}$. In Proposition 4.1 of [BCLP], Béguin, Crovisier, Le Roux and Patou proved the following result:

"Let γ be an essential line in the annulus $\mathbb{A}$. Assume that some lift Γ of γ is disjoint from its images under the maps Φ and Ψ. Then the $q + q' - 1$ first iterates of γ under F are pairwise disjoint, and ordered as the $q + q' - 1$ first iterates of a vertical line under a rigid rotation of angle $\rho \in]\frac{p}{q}, \frac{p'}{q'}[$."

Remark here that the proof of [BCLP] is written in the context of the closed annulus, but it also works in the open annulus setting.

Therefore, to prove the theorem, it is enough to find an essential line γ in $\mathbb{A}$, and a lift of γ which is disjoint from its images under Φ and Ψ.

By the properties of rotation number, we have that $\mathrm{Cl}(\mathrm{Rot}(F^{qq'} \circ T^{-pq'})) \subset]0, 1[$. By Lemma 7, $f^{qq'}$ satisfies the intersection property since f satisfies the property. By Corollary 2 that we have proved above, there exists an oriented essential line γ' in $\mathbb{A}$ that is lifted to an essential line Γ' in $\mathbb{R}^2$ which satisfies

$$\Gamma' < (F^{qq'} \circ T^{-pq'})(\Gamma') < T(\Gamma'). \tag{3.0.1}$$

In particular, we have that $\Gamma' < (F^q \circ T^{-p})^{q'}(\Gamma') = \Phi^{q'}(\Gamma')$ and $\Gamma' < T(\Gamma')$. Acting $T^{pq'-qp'}$ on the formula (3.0.1) and observing that $qp' - pq' = 1$, we get

$$T^{-1}(\Gamma') < (F^{qq'} \circ T^{-qp'})(\Gamma') < \Gamma'.$$

In particular, we have that $\Gamma' < (F^{-q'} \circ T^{p'})^q(\Gamma') = \Psi^q(\Gamma')$.

By Lemma 6, let $H_1 = \Phi$, $H_2 = \Psi$, $H_3 = T$, $q_1 = q'$, $q_2 = q$ and $q_3 = 1$, we can construct an essential line Γ'' in $\mathbb{R}^2$ which satisfies $\Gamma'' < \Phi(\Gamma'')$, $\Gamma'' < \Psi(\Gamma'')$ and $\Gamma'' < T(\Gamma'')$. Therefore, the essential line $\gamma'' = \pi(\Gamma'')$ satisfies the conclusion of the theorem. We have completed the proof. $\square$

4. PROOFS OF THE LEMMAS

Proof of Lemma 6. Let us say that the essential line Γ is of type $(q_1, q_2, \ldots, q_p)$ if $\Gamma < H_i^{q_i}(\Gamma)$ for every $i \in \{1, \ldots, p\}$. We want to prove that the existence of an essential line Γ of type $(q_1, q_2, \ldots, q_p)$ implies the existence of an essential line Γ' of type $(1, \ldots, 1)$. By a simple induction argument, it is sufficient to prove the existence of an essential line Γ' of type $(1, q_2, \ldots, q_p)$.

We choose some essential lines of $(\Gamma_i)_{0 \le i \le q_1 - 1}$ in $\mathbb{R}^2$ such that

$$\Gamma = \Gamma_0 < \Gamma_1 < \cdots < \Gamma_{q_1-1} < H_j^{q_j}(\Gamma_0), \quad \text{for every } j \in \{1, \ldots, p\}.$$

Consider the essential line

$$\Gamma' = H_1^{q_1}(\Gamma_0) \vee H_1^{q_1-1}(\Gamma_1) \vee \cdots \vee H_1(\Gamma_{q_1-1}) = \bigvee_{i=0}^{q_1-1} H_1^{q_1-i}(\Gamma_i).$$

For every $i \in \{0, \ldots, q_1 - 2\}$, we have $\Gamma' \le H_1^{q_1-i}(\Gamma_i)$ (by the definition of Γ' and by item (2) of remark 1) and $H_1^{q_1-i}(\Gamma_i) < H_1^{q_1-i}(\Gamma_{i+1})$. Hence for every $i \in \{0, \ldots, q_1 - 2\}$, we get $\Gamma' < H_1^{q_1-i}(\Gamma_{i+1})$. Moreover, we have $\Gamma' \le H_1(\Gamma_{q_1-1})$ and $\Gamma_{q_1-1} < H_1^{q_1}(\Gamma_0)$. Hence, $\Gamma' < H_1^{q_1+1}(\Gamma_0)$. Finally, using item (1) of remark 1, we get

$$\Gamma' < \bigvee_{i=0}^{q_1-1} H_1^{q_1-i+1}(\Gamma_i) = H_1(\Gamma').$$

Observe that $\Gamma_i < H_j^{q_j}(\Gamma_i)$ for every $i \in \{0, 1, \cdots, q_1 - 1\}$ and $j \in \{2, \ldots, p\}$. Therefore, we have $\Gamma' \le H_1^{q_1-i}(\Gamma_i) < H_1^{q_1-i}(H_j^{q_j}(\Gamma_i))$. Using the definition of Γ', the fact that H_j commutes with H_1, and remark 1, we have

$$\Gamma' < \bigvee_{i=0}^{q-1} H_1^{q_1-i}(H_j^{q_j}(\Gamma_i)) = H_j^{q_j}(\Gamma').$$

Hence, the essential line Γ' is of type $(1, q_2, \ldots, q_p)$. We have completed the proof. $\square$

Proof of Lemma 7. It is similar to the proof of Lemma 6. For every essential circle γ in $\mathbb{A}$, we denote by $B(\gamma)$ the connected component of $\mathbb{A} \setminus \gamma$ which is "below γ" (that is, "containing" the lower end S). Given two essential circles γ and γ' in $\mathbb{A}$, the boundary of

the connected component of $B(\gamma) \cap B(\gamma')$ "containing" S is an essential circle, which we denote by $\gamma \vee \gamma'$. We write $\gamma < \gamma'$ if $\mathrm{Cl}(B(\gamma)) \subset B(\gamma')$.

To prove by contraction, we suppose that there exists a positive integer q and an essential circle γ such that $\gamma < f^q(\gamma)$ (by replacing f by f^{-1} if necessary). We consider some essential circles $(\gamma_i)_{0 \leq i \leq q-1}$ such that

$$\gamma = \gamma_0 < \gamma_1 < \cdots < \gamma_{q-2} < \gamma_{q-1} < f^q(\gamma_0)$$

and we define

$$\gamma' = \bigvee_{i=0}^{q-1} f^{q-i}(\gamma_i).$$

Like in the proof of Lemma 6, we get $\gamma' < f(\gamma')$. In particular, we obtain an essential circle γ' that is disjoint from its image under f. $\square$

To prove Lemma 8, we need the following lemma:

Lemma 9. *Let $f \in \mathrm{Homeo}_*^{\wedge}(\mathbb{A})$ and F be a lift of f to $\mathbb{R}^2$. Suppose that there exist two path connected sets X_1 and X_2 in $\mathbb{R}^2$ satisfying*

1. *$F(X_i) \cap X_i = \emptyset$ for $i = 1, 2$;*
2. *$T^k(X_i) \cap X_i = \emptyset$ ($i = 1, 2$) for every $k \in \mathbb{Z} \setminus \{0\}$;*
3. *Either $X_1 = X_2$ or $T^k(X_1) \cap X_2 = \emptyset$ for every $k \in \mathbb{Z}$;*
4. *There exist positive integers p_i, q_i ($i = 1, 2$) such that*

$$F^{q_1}(X_1) \cap T^{p_1}(X_1) \neq \emptyset \quad \text{and} \quad F^{q_2}(X_2) \cap T^{-p_2}(X_2) \neq \emptyset.$$

Then F has a fixed point.

Proof. Since $F^{q_1}(X_1) \cap T^{p_1}(X_1) \neq \emptyset$ and $F^{q_2}(X_2) \cap T^{-p_2}(X_2) \neq \emptyset$, there exist points $x_i, x_i' \in X_i$ ($i = 1, 2$) such that $F^{q_1}(x_1) = T^{p_1}(x_1')$ and $F^{q_2}(x_2) = T^{-p_2}(x_2')$. In the case where $X_1 \neq X_2$, we choose a segment Γ_1 (resp. Γ_2) of X_1 (resp. X_2) that contains x_1 and x_1' (resp. x_2 and x_2'). In the case where $X_1 = X_2$, we choose a finite tree included in X_1 which contains x_i, x_i' ($i = 1, 2$) and we write $\Gamma_1 = \Gamma_2$ for this tree. Then we can replace the couple (X_1, X_2) by the couple (Γ_1, Γ_2) which satisfies the same property. The items (2) and (3) of Lemma 9 imply that $\pi(\Gamma_1)$ and $\pi(\Gamma_2)$ are two disjoint segments or a finite tree in $\mathbb{A}$.

To prove by contradiction, we suppose that F has no fixed point. We can find a maximal free brick decomposition $\mathfrak{B}$ of the plane that is T-invariant such that B_1 and B_2 are bricks of this decomposition and contain Γ_1 and Γ_2 respectively (note here that $B_1 = B_2$ if $X_1 = X_2$). We have that $B_1 \prec T^{p_1}(B_1)$ and $B_2 \prec T^{-p_2}(B_2)$ by the hypotheses where $B \prec B'$ means $B \preceq B'$ but $B' \npreceq B$. This implies that, for every $n, n' \geq 1$, we have

$$B_1 \prec T^{np_1}(B_1) \quad \text{and} \quad B_2 \prec T^{-n'p_2}(B_2). \tag{4.0.2}$$

Now consider the sets

$$\bigcup_{k \in \mathbb{Z}} T^k(B_{1\succeq}) = \bigcup_{k \in \mathbb{Z}} T^k(B_1)_{\succeq} \quad \text{and} \quad \bigcup_{k \in \mathbb{Z}} T^k(B_{2\succeq}) = \bigcup_{k \in \mathbb{Z}} T^k(B_2)_{\succeq}.$$

We claim that the inclusions $B_2 \subset \bigcup_{k\in\mathbb{Z}} T^k(B_{1\succeq})$ and $B_1 \subset \bigcup_{k\in\mathbb{Z}} T^k(B_{2\succeq})$ can not happen simultaneously. Otherwise, there exist integers k and k', satisfying that

$$T^k(B_1) \preceq B_2 \quad \text{and} \quad T^{k'}(B_2) \preceq B_1. \tag{4.0.3}$$

From (4.0.2) and (4.0.3), we get $T^{k+n'p_2+k'}(B_1) \prec B_1$ for every $n' \geq 1$. When n' is large enough we have $k'' = k + k' + n'p_2 > 0$. Hence, we have $T^{k''p_1}(B_1) \prec B_1$. By (4.0.2), we also have $B_1 \prec T^{k''p_1}(B_1)$. It implies that $B_1 \prec B_1$, which is impossible. The fact that the inclusions $B_2 \subset \bigcup_{k\in\mathbb{Z}} T^k(B_{1\succeq})$ and $B_1 \subset \bigcup_{k\in\mathbb{Z}} T^k(B_{2\succeq})$ can not happen simultaneously implies that

$$\bigcup_{k\in\mathbb{Z}} T^k(B_{1\succeq}) \cap \bigcup_{k\in\mathbb{Z}} T^k(B_{2\preceq}) = \emptyset \quad \text{or} \quad \bigcup_{k\in\mathbb{Z}} T^k(B_{1\preceq}) \cap \bigcup_{k\in\mathbb{Z}} T^k(B_{2\succeq}) = \emptyset.$$

As $T^{p_1}(B_1) \in B_{1\succeq}$ and $T^{p_2}(B_2) \in B_{2\preceq}$, we know that $\bigcup\limits_{k\in\mathbb{Z}} T^k(B_{1\succeq})$ and $\bigcup\limits_{k\in\mathbb{Z}} T^k(B_{2\preceq})$ are connected. This implies they project by π onto essential connected surfaces (i.e. containing an essential circle) $b_{1\succeq}$ and $b_{2\preceq}$ with boundaries. So there exists at least a connected component of the boundary of $b_{1\succeq}$ that is an essential circle and is free for f, which is contrary to the fact f has the intersection property. We can also get a contradiction in the case where $\bigcup_{k\in\mathbb{Z}} T^k(B_{1\preceq}) \cap \bigcup_{k\in\mathbb{Z}} T^k(B_{2\succeq}) = \emptyset$.

Hence, F has at least one fixed point, we complete the proof. □

Proof of Lemma 8. We know that the map f satisfies the intersection property, that the lift $F \circ T^{q-1}$ is fixed point free, and that there is a positive recurrent point of f with positive rotation number for this new lift. By Theorem 4, there is an oriented essential line γ of $\mathbb{A}$ joining S to N that is lifted to an essential Brouwer line Γ of $F \circ T^{q-1}$ which satisfies

$$\Gamma < (F \circ T^{q-1})(\Gamma).$$

We can write $\Gamma < T(\Gamma) < (F \circ T^q)(\Gamma)$. Write $F' = F \circ T^q$, then we have

$$F'^{-n}(\Gamma) < T^{-n}(\Gamma) < \Gamma < T^n(\Gamma) < F'^n(\Gamma) \quad \text{for every} \quad n \geq 1,$$

which implies that F' is conjugate to a translation. We consider the annulus $\mathbb{A}' = \mathbb{R}^2/F'$ and the homeomorphism t induced by T on $\mathbb{A}'$. The map $F^{-1} \circ T^q = F'^{-1} \circ T^{2q}$ is a lift of t^{2q} that is fixed point free. By Theorem 4, there are three cases to consider:

(1) There exists an oriented essential line γ' of $\mathbb{A}'$ that is lifted to an oriented line Γ' in $\mathbb{R}^2$ such that $\Gamma' < F'(\Gamma')$ and $\Gamma' < (F^{-1} \circ T^q)(\Gamma')$;
(2) There exists an oriented essential line γ' of $\mathbb{A}'$ that is lifted to an oriented line Γ' in $\mathbb{R}^2$ such that $\Gamma' < F'(\Gamma')$ and $(F^{-1} \circ T^q)(\Gamma') < \Gamma'$;
(3) There exists an essential circle in $\mathbb{A}'$ that is free for t^{2q}.

We will get the lemma in the first case and contradictions in the two other cases.

In the case (1), we have

$$T^{-q}(\Gamma') < F(\Gamma') < T^q(\Gamma'). \tag{4.0.4}$$

Applying T^q to (4.0.4), we get $\Gamma' < F'(\Gamma') < T^{2q}(\Gamma')$. Thus we have

$$T^{-2qn}(\Gamma') < F'^{-n}(\Gamma') < \Gamma' < F'^n(\Gamma') < T^{2qn}(\Gamma') \quad \text{for every} \quad n \geq 1. \tag{4.0.5}$$

Observe that $\bigcup_{n\geq 1} R(F'^{-n}(\Gamma'))\cap L(F'^{n}(\Gamma'))=\mathbb{R}^2$ since F' is conjugate to a translation and the line $\gamma'=\pi(\Gamma')$ is an essential line in $\mathbb{A}'$. For any $P>0$, by the compactness of $[-q,q]\times[-P,P]$, there exists a positive integer N such that $[-q,q]\times[-P,P]\subset R(T^{-2Nq}(\Gamma'))\cap L(T^{2Nq}(\Gamma'))$. We deduces that $[-q,+\infty[\times[-P,P]\subset R(T^{-2Nq}(\Gamma'))$ and $]-\infty,q]\times[-P,P]\subset L(T^{2Nq}(\Gamma'))$, and hence $[-q+2Nq,+\infty[\times[-P,P]\subset R(\Gamma')$ and $]-\infty,q-2Nq[\times[-P,P]\subset L(\Gamma')$. This implies that Γ' is essential.

By Lemma 6, let $H_1=T$, $H_2=T^q\circ F^{-1}$, $H_3=T^q\circ F$, $q_1=2q$ and $q_2=q_3=1$, we can construct an essential line Γ'' which satisfies

$$T^{-q}(\Gamma'')<F(\Gamma'')<T^q(\Gamma'')$$

and

$$\Gamma''<T(\Gamma'').$$

Hence, $\gamma''=\pi(\Gamma'')$ is an essential line of $\mathbb{A}$ that satisfies the conclusion of the lemma.

In the case (2), we will see how to get a contradiction.

The line Γ' satisfies $\Gamma'<(F\circ T^q)(\Gamma')$ and $\Gamma'<(F\circ T^{-q})(\Gamma')$. We define the set X of couple of integers (m,n) such that $\Gamma'<(F^m\circ T^n)(\Gamma')$. It is a set stable by addition that contains $(1,q)$ and $(1,-q)$. So it contains all the integers $(m+n,q(m-n))$, where $m\geq 0$, $n\geq 0$ and at least one is non zero. This is exactly the set of couples (m,nq) where $m>0$ and $|n|\leq m$. For every couple (m,n) of integers such that $m>0$ and $|\frac{n}{m}|\leq q-\frac{2}{m}$, we have $qm-1>0$ and $|n-1|\leq|n|+1\leq qm-1$. Therefore, we have $q(m,n)-(1,q)\in X$, which means

$$\Gamma'<(F\circ T^q)(\Gamma')<(F^m\circ T^n)^q(\Gamma'). \tag{4.0.6}$$

Fix $z\in \mathrm{Rec}^+(f)$ having a rotation number ρ and $\widetilde{z}\in\pi^{-1}(z)$. Since $F'=F\circ T^q$ is conjugate to a translation and the line γ' is an essential line in $\mathbb{A}'$, we can always suppose that $\widetilde{z}$ is contained in the region $\mathrm{Cl}(R(\Gamma'))\cap L((F\circ T^q)(\Gamma'))$ by replacing Γ' with an iterate $F'^k(\Gamma')$ if necessary. Consider the homeomorphism f_q of the annulus $\mathbb{A}_q=\mathbb{R}^2/T^q$ lifted by F and write $\pi_q:\mathbb{R}^2\to\mathbb{A}_q$ for the covering map. We know that $\mathrm{Rec}^+(f)=\mathrm{Rec}^+(f^q)$ and we can prove similarly that $\pi_q(\widetilde{z})\in\mathrm{Rec}^+(f_q^q)$ (we give a proof in the Appendix, see Lemma 19). In other words, there exist two sequences of integers $(n_i)_{i\geq1}$ and $(m_i)_{i\geq1}$ such that $m_i\to+\infty$ and $F^{qm_i}\circ T^{qn_i}(\widetilde{z})\to\widetilde{z}$ as $i\to+\infty$. Certainly, $\frac{qn_i}{qm_i}\to-\rho$ as $i\to+\infty$. Therefore, there is a positive integer N such that when $i\geq N$, we have $|\frac{n_i}{m_i}|<q-1<q-\frac{2}{m_i}$.

By the inequation (4.0.6), we have $\Gamma'<(F\circ T^q)(\Gamma')<(F^{m_i}\circ T^{n_i})^q(\Gamma')$ when $i\geq N$. On one hand, the points of the sequence $\{(F^{m_i}\circ T^{n_i})^q(\widetilde{z})\}_{i\geq N}$ belong to $R(F\circ T^q(\Gamma'))$ and so their limit belongs to $\mathrm{Cl}(R(F\circ T^q(\Gamma')))$. On the other hand, the limit of the sequence belongs to $L(F\circ T^q(\Gamma'))$, which is a contradiction.

In the case (3), we will get a contradiction again.

By Lemma 7, we deduce that there exists an essential circle γ' free for t. Therefore γ' lifts to a Brouwer line Γ' for T, in particular, $\Gamma'\cap T(\Gamma')=\emptyset$.

The curve γ' is closed in $\mathbb{A}'$, which implies that its lift Γ' satisfies

$$\Gamma' = F'(\Gamma') = (F \circ T^q)(\Gamma').$$

Hence

$$T^{-q}(\Gamma') = F(\Gamma') \quad \text{and} \quad F(\Gamma') \cap T^{1-q}(\Gamma') = \emptyset.$$

Considering the map $F'' = F \circ T^{q-1}$, we have the following:

- there exists a free line Γ' of $\mathbb{R}^2$ for F'' and for T (and hence for T^k for every $k \neq 0$) such that $T^{-1}(\Gamma') = F''(\Gamma')$;
- $\emptyset \neq \mathrm{Rot}(F'') \subset [1, 2q-3]$, in particular, F'' has no fixed point.

The first item above implies that there exists a segment $\Gamma_0 \subset \Gamma'$ such that $F''(\Gamma_0) \cap T^{-1}(\Gamma_0) \neq \emptyset$ and that Γ_0 is free both for F'' and for T^k for any $k \in \mathbb{Z} \setminus \{0\}$.

The second item above implies that there exists $z \in \mathbb{A}$ such that $z \in \mathrm{Rec}^+(f)$ and $\rho(F''; z) \in [1, 2q-3]$. We claim that $z \notin \pi(\Gamma_0)$. Otherwise, choose a lift $\widetilde{z}$ of z in Γ_0. Let U be an open disk containing $\widetilde{z}$, located in the region between $T^{-1}(\Gamma')$ and $T(\Gamma')$. There exist positive integers $n \geq 2$, $l \geq 1$ such that $F''^n(\widetilde{z}) \in T^l(U)$. As we have $F''^n(\widetilde{z}) \in F''^n(\Gamma_0) \subset T^{-n}(\Gamma')$, we deduce that $T^l(U) \cap T^{-n}(\Gamma') \neq \emptyset$ which is impossible. We have completed the claim. As $z \notin \pi(\Gamma_0)$, we can find a disk U' free for F'', containing $\widetilde{z}$, disjoint from Γ_0, and satisfying $T^k(U') \cap U' = \emptyset$ for every $k \neq 0$ and $F''^n(U') \cap T^l(U') \neq \emptyset$.

By Lemma 9, F'' has a fixed point, which is impossible. $\square$

In the rest of this section, we give some remarks about the relations between the positively recurrent set and the rotation number set.

Let $f \in \mathrm{Homeo}_*(\mathbb{A})$ and F be a lift of f to $\mathbb{R}^2$. Suppose that $z \in \mathrm{Rec}^+(f)$ and $\widetilde{z} \in \pi^{-1}(z)$. We define $\mathcal{E}(z) \subset \mathbb{R} \cup \{-\infty, +\infty\}$ by saying that $\rho \in \mathcal{E}(z)$ if there exists a sequence $\{n_k\}_{k=1}^{+\infty} \subset \mathbb{N}$ such that

- $\lim\limits_{k \to +\infty} f^{n_k}(z) = z$;
- $\lim\limits_{k \to +\infty} \dfrac{p_1(F^{n_k}(\widetilde{z})) - p_1(\widetilde{z})}{n_k} = \rho.$

Define $\rho^-(F; z) = \inf \mathcal{E}(z)$ and $\rho^+(F; z) = \sup \mathcal{E}(z)$. Obviously, we have that $\rho(F; z)$ exists if and only if $\rho^-(F; z) = \rho^+(F; z) \in \mathbb{R}$.

The following proposition is due to Franks [F1] when $\mathbb{A}$ is closed annulus and f has no wandering point, and it was improved by Le Calvez [Lec2] when $\mathbb{A}$ is open annulus and f satisfies the intersection property. We can use Lemma 9 to prove it.

Proposition 10. *Let $f \in \mathrm{Homeo}_*^{\wedge}(\mathbb{A})$ and F be a lift of f to $\mathbb{R}^2$. Suppose that there exist two recurrent points z_1 and z_2 such that $-\infty \leq \rho^-(F; z_1) < \rho^+(F; z_2) \leq +\infty$. Then for any rational number $p/q \in]\rho^-(F; z_1), \rho^+(F; z_2)[$ written in an irreducible way, there exists a periodic point of period q whose rotation number is p/q.*

Proof. We consider the map f^q and its lift $F' = F^q \circ T^{-p}$. It is sufficient to prove that F' has a fixed point.

By Lemma 7, f^q satisfies the intersection property. By the properties of the rotation number, we have $\rho^-(F'; z_1) < 0 < \rho^+(F'; z_2)$. For any fixed point z of f^q, we have

$\rho^-(F';z) = \rho^+(F';z) \in \mathbb{Z}$. Therefore, the points z_1 and z_2 are not fixed points of f^q. So we can choose open disks b_1 and b_2, containing respectively z_1 and z_2, equal if $z_1 = z_2$, disjoint if $z_1 \neq z_2$, such that the connected components of $\pi^{-1}(b_1)$ and $\pi^{-1}(b_2)$ are free for F'. Choose a component $\widetilde{b}_1$ of $\pi^{-1}(b_1)$ and a component $\widetilde{b}_2$ of $\pi^{-1}(b_2)$ (equal to $\widetilde{b}_1$ if $z_1 = z_2$). Observe that $\widetilde{b}_1$ and $\widetilde{b}_2$ satisfy all the hypotheses of Lemma 9. Then, Proposition 10 follows from Lemma 9. $\square$

From the Lemma 9 and Proposition 10, we have the following corollaries:

Corollary 11. *Let $f \in \mathrm{Homeo}_*^{\wedge}(\mathbb{A})$ and F be a lift of f to $\mathbb{R}^2$. If $\mathrm{Rec}^+(f) \neq \emptyset$, then $\mathrm{Rot}(F) \neq \emptyset$.*

Proof. Let $z \in \mathrm{Rec}^+(f)$ and $\widetilde{z}$ be a lift of z. If z is a fixed point, the corollary is true. Otherwise, choose a free disk b for f in $\mathbb{A}$ that contains z and a connected component $\widetilde{b}$ of $\pi^{-1}(b)$.

If $\rho^-(F;z) \neq \rho^+(F;z)$, then $\mathrm{Rot}(F) \neq \emptyset$ by Proposition 10. If $\rho^-(F;z) = \rho^+(F;z) \in \mathbb{R}$, then z has a rotation number and therefore $\mathrm{Rot}(F) \neq \emptyset$. It remains to study the cases $\rho^-(F;z) = \rho^+(F;z) = +\infty$ and $\rho^-(F;z) = \rho^+(F;z) = -\infty$. They are similar. We will explain now why the condition $\rho^-(F;z) = \rho^+(F;z) = +\infty$ implies that f has a fixed point and therefore that $\mathrm{Rot}(F) \neq \emptyset$. We can find two rational numbers $\frac{p_i}{q_i}$ $(i = 1, 2)$ and an integer n, arbitrarily large, such that $F^{q_1}(\widetilde{b}) \cap T^{p_1}(\widetilde{b}) \neq \emptyset$ and $F^{q_2}(\widetilde{b}) \cap T^{p_2}(\widetilde{b}) \neq \emptyset$. Consider the lift $F' = F \circ T^{-n}$ of f to $\mathbb{R}^2$, we have $F'^{q_1}(\widetilde{b}) \cap T^{p_1 - q_1 n}(\widetilde{b}) \neq \emptyset$ and $F'^{q_2}(\widetilde{b}) \cap T^{p_2 - q_2 n}(\widetilde{b}) \neq \emptyset$. Recall that $\widetilde{b}$ is free for F'. By Lemma 9, we have that F' has a fixed point and hence f has a fixed point. We have completed the proof. $\square$

Corollary 12. *Let $f \in \mathrm{Homeo}_*^{\wedge}(\mathbb{A})$ and F a lift of f to $\mathbb{R}^2$. The set $\mathrm{Rot}(F)$ is reduced to a single irrational number ρ if and only if $\mathrm{Rec}^+(f) \neq \emptyset$ and f has no periodic orbit.*

5. Weak rotation number and the line translation theorem

We have supposed previously that there exists a recurrent point. It is natural to wonder if the main theorem is still true without this hypothesis.

In this section, we define the weak rotation set of f which is a generalization of the rotation set defined in Section 1. It was introduced in [F2] in a more generalized framework (see also [Ler2] for a local study). We first give some properties of weak rotation numbers, then we prove the generalization of the line translation theorem.

Fix $f \in \mathrm{Homeo}_*^{\wedge}(\mathbb{A})$ and a lift F. For every compact set K define the set $\mathrm{Rot}_{\mathrm{weak},K}(F)$ in the following way : $r \in [-\infty, +\infty]$ belongs to $\mathrm{Rot}_{\mathrm{weak},K}(F)$ if there exists a sequence $\{\widetilde{z}_k\}_{k\geq 1} \subset \mathbb{R}^2$ and a sequence $\{n_k\}_{k\geq 1}$ of positive integers such that

- $\pi(\widetilde{z}_k) \in K$;
- $\pi(F^{n_k}(\widetilde{z}_k)) \in K$;
- $\lim\limits_{k\to+\infty} n_k = +\infty$;
- $\lim\limits_{k\to+\infty} \dfrac{p_1(F^{n_k}(\widetilde{z}_k)) - p_1(\widetilde{z}_k)}{n_k} = r.$

Define the weak rotation set of F as following

$$\mathrm{Rot}_{\mathrm{weak}}(F) = \bigcup_{K \in \mathrm{C}(\mathbb{A})} \mathrm{Rot}_{\mathrm{weak},K}(F),$$

where $\mathrm{C}(\mathbb{A})$ is the collection of all compact subsets of $\mathbb{A}$.

Proposition 13. *If $f \in \mathrm{Homeo}_*^{\wedge}(\mathbb{A})$, we have the following properties:*

1. $\mathrm{Rot}_{\mathrm{weak},K}(F)$ *is closed;*
2. *If K separates the two ends N and S of $\mathbb{A}$, then $\mathrm{Rot}_{\mathrm{weak},K}(F) \neq \emptyset$. Therefore $\mathrm{Rot}_{\mathrm{weak}}(F) \neq \emptyset$;*
3. $\mathrm{Rot}(F) \subset \mathrm{Rot}_{\mathrm{weak}}(F)$*;*
4. *For every $p \in \mathbb{Z}$ and $q \in \mathbb{Z}$, we have $\mathrm{Rot}_{\mathrm{weak}}(F^q \circ T^p) = q\mathrm{Rot}_{\mathrm{weak}}(F) + p$.*

Proof. (1) For every $n \geq 1$ define a set $R_n \subset \mathbb{R}$ as

$$R_n = \left\{ \frac{p_1 \circ F^n(\widetilde{z}) - p_1(\widetilde{z})}{n} \,\Big|\, \widetilde{z} \in \pi^{-1}(K) \quad \text{and} \quad F^n(\widetilde{z}) \in \pi^{-1}(K) \right\}.$$

Observe now that

$$\mathrm{Rot}_{\mathrm{weak},K}(F) = \bigcap_{n \geq 1} \mathrm{Cl}\Big(\bigcup_{k \geq n} R_k\Big),$$

so $\mathrm{Rot}_{\mathrm{weak},K}(F)$ is closed.

(2) To prove that $\mathrm{Rot}_{\mathrm{weak},K}(F)$ is not empty, it is sufficient to prove that for every $q \geq 1$, there is a point $z_q \in K$ such that $f^q(z_q) \in K$. Suppose that there exists $q \geq 1$ such that $K \cap f^q(K) = \emptyset$. As K is compact, we can find an open neighborhood U of K such that $U \cap f^q(U) = \emptyset$. Since K separates the two ends of $\mathbb{A}$, we deduce that U contains an essential circle. This contradicts Lemma 7.

(3) For every $r \in \mathrm{Rot}(F)$, there exists $z \in \mathrm{Rec}^+(f)$ such that $\rho(F; z) = r$, we choose K to be a closed disk whose interior contains z. By definitions, we have $r = \rho(F; z) \in \mathrm{Rot}_{\mathrm{weak},K}(F) \subset \mathrm{Rot}_{\mathrm{weak}}(F)$. Therefore, $\mathrm{Rot}(F) \subset \mathrm{Rot}_{\mathrm{weak}}(F)$.

(4) Since $F \circ T = T \circ F$, we clearly have $\mathrm{Rot}_{\mathrm{weak}}(F \circ T^p) = \mathrm{Rot}_{\mathrm{weak}}(F) + p$. Let us prove now that $\mathrm{Rot}_{\mathrm{weak}}(F^q) = q\mathrm{Rot}_{\mathrm{weak}}(F)$ for every $q \in \mathbb{Z}$. When $q = 0$, it is trivial. First, fix $q > 0$. Obviously, $\mathrm{Rot}_{\mathrm{weak},K}(F^q) \subset q\mathrm{Rot}_{\mathrm{weak},K}(F)$ for any $K \in \mathrm{C}(\mathbb{A})$ by definition. Therefore, $\mathrm{Rot}_{\mathrm{weak}}(F^q) \subset q\mathrm{Rot}_{\mathrm{weak}}(F)$.

Now, we prove the inverse. For every $r \in \mathrm{Rot}_{\mathrm{weak}}(F)$, there exists $K \in \mathrm{C}(\mathbb{A})$ such that $r \in \mathrm{Rot}_{\mathrm{weak},K}(F)$. Therefore, there exists a sequence $\{\widetilde{z}_k\}_{k \geq 1} \subset \mathbb{R}^2$ and a sequence $\{n_k\}_{k \geq 1}$ of positive integers such that

- $\pi(\widetilde{z}_k) \in K$;
- $\pi(F^{n_k}(\widetilde{z}_k)) \in K$;
- $\lim\limits_{k \to +\infty} n_k = +\infty$;
- $\lim\limits_{k \to +\infty} \dfrac{p_1(F^{n_k}(\widetilde{z}_k)) - p_1(\widetilde{z}_k)}{n_k} = r.$

Write $n_k = l_k q + p_k$ where $0 \le p_k < q$. Suppose that there are infinitely many k such that $p_k = p$ where $0 \le p < q$. We can choose a subsequence $\{n_{k_j}\}_{j\ge1}$ of $\{n_k\}_{k\ge1}$ such that $n_{k_j} = l_{k_j} q + p$ and

$$\lim_{j\to+\infty} \frac{p_1(F^{ql_{k_j}}(F^p(\widetilde{z}_{k_j}))) - p_1(F^p(\widetilde{z}_{k_j}))}{ql_{k_j}} = r.$$

Then we have

$$qr = \lim_{j\to+\infty} \frac{p_1(F^{ql_{k_j}}(F^p(\widetilde{z}_{k_j}))) - p_1(F^p(\widetilde{z}_{k_j}))}{l_{k_j}} \in \mathrm{Rot}_{\mathrm{weak},f^p(K)}(F^q).$$

Hence $q\mathrm{Rot}_{\mathrm{weak}}(F) \subset \mathrm{Rot}_{\mathrm{weak}}(F^q)$.

For the case $q < 0$, it is sufficient to prove that $\mathrm{Rot}_{\mathrm{weak},K}(F^{-1}) = -\mathrm{Rot}_{\mathrm{weak},K}(F)$ for every $K \in \mathrm{C}(\mathbb{A})$. For every $r \in \mathrm{Rot}_{\mathrm{weak},K}(F)$, by letting $\widetilde{z}'_k = F^{n_k}(\widetilde{z}_k)$ in the definition of $\mathrm{Rot}_{\mathrm{weak},K}(F)$, we have

- $\pi(\widetilde{z}'_k) \in K$;
- $\pi(F^{-n_k}(\widetilde{z}'_k)) \in K$;
- $\lim\limits_{k\to+\infty} n_k = +\infty$;
- $\lim\limits_{k\to+\infty} \dfrac{p_1(F^{-n_k}(\widetilde{z}'_k)) - p_1(\widetilde{z}'_k)}{n_k} = -r.$

Therefore,

$$-\mathrm{Rot}_{\mathrm{weak},K}(F) \subset \mathrm{Rot}_{\mathrm{weak},K}(F^{-1}). \tag{5.0.7}$$

By replacing F by F^{-1} in the conclusion (5.0.7), we have $\mathrm{Rot}_{\mathrm{weak},K}(F^{-1}) \subset -\mathrm{Rot}_{\mathrm{weak},K}(F)$. We have completed the proof. $\square$

Lemma 14. *Suppose that there exists a compact set $K \subset \mathbb{A}$ and an orientated line Γ of $\mathbb{R}^2$ satisfying*

(1) $\Gamma < T(\Gamma)$*;*
(2) $\Gamma < F(\Gamma)$ *(resp. $F(\Gamma) < \Gamma$);*
(3) $\pi^{-1}(K) \subset \bigcup_{k\in\mathbb{Z}} T^k(\mathrm{Cl}(R(\Gamma)) \cap L(T(\Gamma)))$.

Then $\mathrm{Rot}_{\mathrm{weak},K}(F) \subset [0,+\infty]$ (resp. $\mathrm{Rot}_{\mathrm{weak},K}(F) \subset [-\infty,0]$). As a consequence, if Γ is a lift of an oriented essential line in $\mathbb{A}$ joining S to N, satisfying $\Gamma < F(\Gamma)$ (resp. $F(\Gamma) < \Gamma$), then $\mathrm{Rot}_{\mathrm{weak}}(F) \subset [0,+\infty]$ (resp. $\mathrm{Rot}_{\mathrm{weak}}(F) \subset [-\infty,0]$).

Proof. We will make a proof by contradiction, we suppose that $\Gamma < T(\Gamma)$ and that there exists a negative number ρ such that $\rho \in \mathrm{Rot}_{\mathrm{weak},K}(F)$. Write

$$\widetilde{K} = \pi^{-1}(K) \cap \mathrm{Cl}(R(\Gamma)) \cap L(T(\Gamma))$$

and

$$\mathrm{width}(\widetilde{K}) = \sup_{\widetilde{z}_1,\widetilde{z}_2\in\widetilde{K}} \{p_1(\widetilde{z}_1) - p_1(\widetilde{z}_2)\}.$$

We first claim that $\mathrm{width}(\widetilde{K})$ is finite. According to the hypothesis (3), we have $K = \pi(\widetilde{K})$. For every $k \geq 1$, we define an open set

$$\widetilde{U}_k = \big(R(T^{-1}(\Gamma)) \cap L(T(\Gamma))\big) \cap \{(x,y) \in \mathbb{R}^2 \mid -k < x < k\}.$$

By the hypothesis (1), the sequence of open sets $\{\pi(\widetilde{U}_k)\}_{k\geq 1}$ in $\mathbb{A}$ is increasing. Since $\mathrm{Cl}(R(\Gamma))\cap L(T(\Gamma)) \subset R(T^{-1}(\Gamma))\cap L(T(\Gamma))$, we have $\widetilde{K}\cap\{(x,y) \in \mathbb{R}^2 \mid -k < x < k\} \subset \widetilde{U}_k$. Obviously, $\widetilde{K} \subset \bigcup_{k\geq 1} \widetilde{U}_k$ and $K \subset \bigcup_{k\geq 1} \pi(\widetilde{U}_k)$. As K is compact, there is a positive integer N such that $K \subset \pi(\widetilde{U}_N)$. It implies that, for every $\widetilde{z} \in \widetilde{K}$, there exists $k \in \mathbb{Z}$ such that $T^k(\widetilde{z}) \in \widetilde{U}_N$. Observe that

$$T^l(R(T^{-1}(\Gamma)) \cap L(T(\Gamma))) \cap (\mathrm{Cl}(R(\Gamma)) \cap L(T(\Gamma))) = \emptyset \quad \text{if} \quad l \in \mathbb{Z} \setminus \{0,1\}.$$

It implies that the only possibilities for k are -1 and 0. So $\widetilde{K} \subset \widetilde{U}_N \cup T(\widetilde{U}_N)$. It completes the claim.

By the definition of $\mathrm{Rot}_{\mathrm{weak},K}(F)$, there is a sequence $\{\widetilde{z}_k\}_{k\geq 1} \subset \widetilde{K}$ and a sequence $\{n_k\}_{k\geq 1}$ of positive integers such that

- $\lim\limits_{k\to+\infty} n_k = +\infty$;
- $\{\pi(F^{n_k}(\widetilde{z}_k))\}_{k\geq 1} \subset K$;
- $\lim\limits_{k\to+\infty} \dfrac{p_1(F^{n_k}(\widetilde{z}_k)) - p_1(\widetilde{z}_k)}{n_k} = \rho$.

If k is large enough, then

$$p_1(F^{n_k}(\widetilde{z}_k)) - p_1(\widetilde{z}_k) < -2\mathrm{width}(\widetilde{K}).$$

It implies that $F^{n_k}(\widetilde{z}_k)$ is on the left of Γ. But, by the hypothesis (2), for every $n \geq 1$ and $\widetilde{z} \in \widetilde{K}$, $F^n(\widetilde{z})$ is on the right of Γ, we get a contradiction.

Let us prove now the second statement of the lemma. The line Γ is a lift of an oriented essential line in $\mathbb{A}$ joining S to N. Observe that $\bigcup_{k\in\mathbb{Z}} T^k(\mathrm{Cl}(R(\Gamma)) \cap L(T(\Gamma))) = \mathbb{R}^2$, then we get the last consequence. □

Using Lemma 14, we can get more properties about the weak rotation number set of F.

Proposition 15. *We have the following properties:*

(1) *If $\frac{p}{q}$ is given and if $\mathrm{Rot}_{\mathrm{weak},K}(F)$ contains r_1 and r_2 where $r_1 < \frac{p}{q} < r_2$, then there exists $\widetilde{z}$ such that $F^q(\widetilde{z}) = T^p(\widetilde{z})$.*

(2) *The sets $\mathrm{Cl}(\mathrm{Rot}_{\mathrm{weak}}(F))$ and $\mathrm{Cl}(\mathrm{Rot}(F))$ are the same intervals except in the case where $\mathrm{Rec}^+(f) = \emptyset$ and $\mathrm{Rot}_{\mathrm{weak}}(F)$ is reduced to an element of $\mathbb{R}$.*

Proof. To prove (1), consider the map f^q and its lift $F' = F^q \circ T^{-p}$. We must prove that F' has a fixed point. We suppose that F' has no fixed point. By Lemma 7, f^q satisfies the intersection property. By Theorem 4, there exists an essential line of $\mathbb{A}$ joining S to N that lifts to a Brouwer line Γ of F'. Then either $\Gamma < F'(\Gamma)$ or $F'(\Gamma) < \Gamma$. By Lemma 14, we have either $\mathrm{Rot}_{\mathrm{weak}}(F) \subset [0,+\infty]$ or $\mathrm{Rot}_{\mathrm{weak}}(F) \subset [-\infty, 0]$. However, by the assertion (4) of Proposition 13, we have $qr_1 - p \in \mathrm{Rot}_{\mathrm{weak}\,K}(F')$ and $qr_2 - p \in \mathrm{Rot}_{\mathrm{weak}\,K}(F')$ with $qr_1 - p < 0 < qr_2 - p$, which is impossible. Therefore, F' has a fixed point.

By the assertion (3) of Proposition 13, we know that the inclusion $\mathrm{Cl}(\mathrm{Rot}(F)) \subset \mathrm{Cl}(\mathrm{Rot}_{\mathrm{weak}}(F))$ is true. If $\mathrm{Rot}_{\mathrm{weak}}(F)$ is reduced to a real number and if $\mathrm{Rec}^+(f) \neq \emptyset$, then by Corollary 11, $\mathrm{Rot}(F)$ is not empty and therefore is equal to $\mathrm{Rot}_{\mathrm{weak}}(F)$. To get (2), it is sufficient first to prove that $\mathrm{Cl}(\mathrm{Rot}_{\mathrm{weak}}(F)) \subset \mathrm{Cl}(\mathrm{Rot}(F))$ in the case where $\mathrm{Rot}_{\mathrm{weak}}(F)$ is not reduced to a point and then to explain why $\mathrm{Rot}_{\mathrm{weak}}(F)$ can not be reduced to $\{+\infty\}$ or $\{-\infty\}$.

Suppose that $\mathrm{Rot}_{\mathrm{weak}}(F)$ is not reduced to a point and denote by I the interior of its convex hull. According to (1), we have

$$\mathrm{Cl}(\mathrm{Rot}_{\mathrm{weak}}(F)) \subset \mathrm{Cl}(I) = \mathrm{Cl}(I \cap \mathbb{Q}) \subset \mathrm{Cl}(\mathrm{Rot}_{\mathrm{per}}(F)) \subset \mathrm{Cl}(\mathrm{Rot}(F)),$$

where $\mathrm{Rot}_{\mathrm{per}}(F)$ is the set of rotation numbers of periodic points.

Now we turn to explain why $\mathrm{Rot}_{\mathrm{weak}}(F)$ can not be reduced to $\{+\infty\}$ or $\{-\infty\}$.

We state the first case, the other case is similar. Consider the sphere $\mathbf{S}^2 = \mathbb{A} \sqcup \{N, S\}$. Suppose first that there exists a non trivial invariant continuum K that contains the end N but not the end S. Consider the connected component U of $\mathbf{S}^2 \setminus K$ that contains S. It is simply connected (property of plane topology) and invariant by f. We can define the *prime end compactification* of U, introduced by Carathéodory [C], by adding a circle $\mathbf{S}^1$. A very good exposition of the theory of prime ends in modern terminology can be found in the paper [M] of Mather.

The prime end compactification can be defined purely topologically but has another significance if we put a complex structure on $\mathbf{S}^2$. We can find a conformal map ϕ between U and the open disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ and we put on $U \sqcup \mathbf{S}^1$ the topology (up to homeomorphism of the resulting space, is independent of ϕ) induced from the natural topology of $\mathrm{Cl}(\mathbb{D})$ in $\mathbb{C}$ by the bijection

$$\overline{\phi} : U \sqcup \mathbf{S}^1 \to \mathrm{Cl}(\mathbb{D})$$

equal to ϕ on U and to the identity on $\mathbf{S}^1$.

As U is invariant by f, the map $f|_U$ can be extended to a homeomorphism $\bar{f}$ of $U \sqcup \mathbf{S}^1$. Moreover, $\bar{f}$ restricted to the prime ends set $\mathbf{S}^1$ is an orientation preserving homeomorphism of the circle. Therefore, we can define its rotation number, say $\rho \in \mathbb{R}/\mathbb{Z}$.

We take off S from $U \sqcup \mathbf{S}^1$, and then paste the annulus $\mathbf{S}^1 \times [0, +\infty)$ and $U \sqcup \mathbf{S}^1 \setminus \{S\}$ together along $\mathbf{S}^1$. We define now f' as the extension of the map $\bar{f}$ to the new open annulus $\mathbb{A}'$ as follows:

$$f'(x, y) = \begin{cases} \bar{f}(x, y) & \text{if} \quad (x, y) \in U \sqcup \mathbf{S}^1 \setminus \{S\}; \\ (g(x), y) & \text{if} \quad (x, y) \in \mathbf{S}^1 \times [0, +\infty). \end{cases}$$

where g is the map induced by $\bar{f}$ on $\mathbf{S}^1$. By the construction of f', the map f' satisfies the intersection property. Indeed, we suppose that γ is an essential circle in $\mathbb{A}'$. If γ does not meet $S^1 \times [0, +\infty)$, it must meet its image because f satisfies the intersection property. Suppose now that γ meets $S^1 \times [0, +\infty)$. There exists a real number $r \geq 0$ such that γ meets $S^1 \times \{r\}$ but does not meet $S^1 \times (r, +\infty)$. This implies that $f'(\gamma)$ and $f'^{-1}(\gamma)$ do not meet $S^1 \times (r, +\infty)$. As a consequence, one deduces that a point $z \in S^1 \times \{r\}$ cannot be strictly below the circles $f'(\gamma)$ and $f'^{-1}(\gamma)$. This implies that $f'(\gamma)$ meets γ.

For any lift F' of f' to the universal cover of $\mathbb{A}'$, we have the following facts:

- there exists $\rho \in \mathbb{R}$ such that $\mathrm{Rot}_{\mathrm{weak},\mathbf{S}^1\times\{r\}}(F') = \{\rho\}$ for every $r \geq 0$;
- $\mathrm{Rot}_{\mathrm{weak},K'}(F') = \{+\infty\}$ if the set K' is a compact set of $U \setminus \{S\}$ that contains an essential circle.

So the set $\mathrm{Rot}_{\mathrm{weak}}(F')$ is not reduced to a point. The assertion (1) implies that there are many periodic points with a rotation number different from ρ. They correspond to periodic points of our initial map f. We have a contradiction.

This implies that there exists a neighborhood of N that is an *isolating Jordan domain* (see [LecY] for the details). Of course, we have a similar situation for S. Observe that there are no periodic orbits except N and S and that the intersection property guarantees that N and S are neither sinks nor sources, thanks to Le Calvez-Yoccoz's Theorem [LecY], there exists an integer k such that the Lefschetz indices $i(f^k, S)$ and $i(f^k, N)$ are non positive. But because of the Lefschetz formula, we know that the sum is 2 (the Euler characteristic of $\mathbf{S}^2$). We have a contradiction. □

Observe that the proofs of Lemma 6 and Lemma 7 do not use the existence of recurrent points. To adapt the line translation theorem with our new hypothesis, we only need to prove a result similar to Lemma 8.

Lemma 16. *Let $f \in \mathrm{Homeo}_*^{\wedge}(\mathbb{A})$. We suppose that F is a lift of f to $\mathbb{R}^2$ and that*

$$\mathrm{Rot}_{\mathrm{weak}}(F) \subset [-q+2, q-2],$$

where $q \geq 2$, then there exists an oriented essential line γ in $\mathbb{A}$ joining S to N that is lifted to an essential line Γ in $\mathbb{R}^2$ which satisfies

$$T^{-q}(\Gamma) < F(\Gamma) < T^q(\Gamma).$$

Proof. According to the assertion (2) of Proposition 14 and Lemma 8, we only need to consider the case where $\mathrm{Rec}^+(f) = \emptyset$ and $\mathrm{Rot}_{\mathrm{weak}}(F)$ is reduced to a real number ρ.

By the assertion (4) of Proposition 13, we have $\mathrm{Rot}_{\mathrm{weak}}(F \circ T^{q-1}) = \{\rho + q - 1\} \subset [1, 2q-3]$. Of cause, the map $F \circ T^{q-1}$ has no fixed point. We choose a compact set K such that $\mathrm{Rot}_{\mathrm{weak},K}(F \circ T^{q-1}) = \{\rho + q - 1\}$. The map f satisfying the intersection property, by Theorem 4 and Lemma 14, there is an oriented essential line γ of $\mathbb{A}$ joining S to N that is lifted to an essential Brouwer line Γ of $F \circ T^{q-1}$ which satisfies

$$\Gamma < (F \circ T^{q-1})(\Gamma).$$

Like in the proof of Lemma 8, we deduce that $F' = F \circ T^q$ is conjugate to a translation. Here again, we consider the annulus $\mathbb{A}' = \mathbb{R}^2/F'$, the homeomorphism t induced by T on $\mathbb{A}'$ and the lift $F^{-1} \circ T^q = F'^{-1} \circ T^{2q}$ of t^{2q}. As it is fixed point free, here again Theorem 4 asserts that there are three possible cases:

(1) There exists an oriented essential line γ' of $\mathbb{A}'$ that is lifted to an oriented line Γ' in $\mathbb{R}^2$ such that $\Gamma' < F'(\Gamma')$ and $\Gamma' < (F^{-1} \circ T^q)(\Gamma')$;
(2) There exists an oriented essential line γ' of $\mathbb{A}'$ that is lifted to an oriented line Γ' in $\mathbb{R}^2$ such that $\Gamma' < F'(\Gamma')$ and $(F^{-1} \circ T^q)(\Gamma') < \Gamma'$;
(3) There exists an essential circle in $\mathbb{A}'$ that is free for t^{2q}.

In the case (1), the arguments that we gave in the proof of Lemma 8 permit us to construct an essential line of $\mathbb{A}$ that satisfies the conclusion of the lemma.

In the case (2), we will show how to get a contradiction.

The line Γ' satisfies $\Gamma' < (F\circ T^q)(\Gamma')$ and $\Gamma' < (F\circ T^{-q})(\Gamma')$. Here again, the set $X$of couple of integers (m,n) such that $\Gamma' < (F^m\circ T^n)(\Gamma')$ contains the set of couples (m,nq), where $m>0$ and $|n|\leq m$ (see the proof of Lemma 8). Let p be any given positive integer. For every couple (m,n) of integers such that $m>\frac{p}{q}$ and $|\frac{n}{m}|\leq q-\frac{2p}{m}$, we have $qm-p>0$ and $|n-p|\leq|n|+p\leq qm-p$. Therefore, we have $q(m,n)-p(1,q)\in X$, which means

$$\Gamma' < (F\circ T^q)^p(\Gamma') < (F^m\circ T^n)^q(\Gamma'). \tag{5.0.8}$$

Let $\widetilde{K}_q=\pi^{-1}(\bigcup_{0\leq i<q} f^i(K))\cap \mathrm{Cl}(R(T^{-q}(\Gamma))\cap L(\Gamma))$. Since $F'=F\circ T^q$ is conjugate to a translation, γ' is an essential line in $\mathbb{A}'$, we can always suppose that the compact set $\widetilde{K}_q$ is contained in the region $\mathrm{Cl}(R(\Gamma'))\cap L((F\circ T^q)^p(\Gamma'))$ for some positive integer p by replacing Γ' with an iterate $F'^k(\Gamma')$ if necessary. Consider the homeomorphism f_q of the annulus $\mathbb{A}_q=\mathbb{R}^2/T^q$ lifted by F and write $\pi_q:\mathbb{R}^2\to\mathbb{A}_q$ for the covering map. Similarly to the proof of the assertion (4) of Proposition 13, we can find a sequence $\{\widetilde{z}_k\}_{k\geq1}\subset\widetilde{K}_q$, a sequence $\{m_k\}_{k\geq1}$ of positive integers and a sequence $\{n_k\}_{k\geq1}$ of integers such that

- $\lim\limits_{k\to+\infty} m_k=+\infty$;
- $F^{qm_k}(\widetilde{z}_k)\in T^{-qn_k}(\widetilde{K}_q)$ for every $k\geq1$;
- $\dfrac{-qn_k}{qm_k}\in]-q+1,q-1[$ when k is large enough.

Therefore, there is a positive integer N such that when $k\geq N$, we have $m_k>2p>\frac{p}{q}$ and $|\frac{n_k}{m_k}|<q-1<q-\frac{2p}{m_k}$.

By the inequation (5.0.8), we have $\Gamma' < (F\circ T^q)^p(\Gamma') < (F^{m_k}\circ T^{n_k})^q(\Gamma')$ when $k\geq N$. On the one hand, the points of the sequence $\{(F^{m_k}\circ T^{n_k})^q(\widetilde{z}_k)\}_{i\geq N}$ belong to $R((F\circ T^q)^p(\Gamma'))$. On the other hand, the points $(F^{m_k}\circ T^{n_k})^q(\widetilde{z}_k)$ belong to $\widetilde{K}_q\subset L((F\circ T^q)^p(\Gamma'))$, which is a contradiction.

It remains to study the case (3) and try to find a contradiction.

Again, the arguments in the proof of Lemma 8 permit us to construct a line Γ' of $\mathbb{R}^2$ satisfying

- $\Gamma'\cap T(\Gamma')=\emptyset$;
- $T^{-q}(\Gamma')=F(\Gamma')$.

Endow Γ' with the orientation such that $\Gamma'<T(\Gamma')$. As Γ' is a Brouwer line of T, we can find an arc Δ_0 that joins a point $z\in\Gamma'$ to $T(z)$ whose image is between the two lines Γ' and $T(\Gamma')$, and does not intersect the lines but at the extremities. We deduce that $\Delta=\bigcup_{k\in Z}T^k(\Delta_0)$ is the preimage of an essential circle δ of $\mathbb{A}$. We know that $\mathrm{Rot}_{\mathrm{weak},\delta}(F)=\{\rho\}$ (the intersection property of f guarantees that it is not empty). By (4) of Proposition 13 and the properties of Γ', the map $F''=F\circ T^{q-1}$ satisfies the following properties:

- $F''(\Gamma') = T^{-1}(\Gamma') < \Gamma'$;
- $\Delta \subset \bigcup_{k\in\mathbb{Z}} T^k(\mathrm{Cl}(R(\Gamma)) \cap L(T(\Gamma)))$;
- $\mathrm{Rot}_{\mathrm{weak},\delta}(F'') = \{\rho + q - 1\} \subset [1, 2q-3]$.

By Lemma 14, the first two items imply that $\mathrm{Rot}_{\mathrm{weak},\delta}(F'') \subset [-\infty, 0]$, which is contrary to the third item. We have completed the proof. □

Replacing the notation $\mathrm{Cl}(\mathrm{Rot}(F))$ in the proof of Theorem 1 by the notation $\mathrm{Rot}_{\mathrm{weak}}(F)$, we have the following similar theorem.

Theorem 17 (Generalization of the line translation theorem*). *Let $f \in \mathrm{Homeo}_*^{\wedge}(\mathbb{A})$ and F be a lift of f to $\mathbb{R}^2$. Assume that $\mathrm{Rot}_{\mathrm{weak}}(F)$ is contained in a Farey interval $]\frac{p}{q}, \frac{p'}{q'}[$. Then, there exists an essential line γ in $\mathbb{A}$ such that the lines $\gamma, f(\gamma), \cdots, f^{q+q'-1}(\gamma)$ are pairwise disjoint. Moreover, the cyclic order of these lines is the same as the cyclic order of the $q+q'-1$ first iterates of a vertical line $\{\theta\} \times \mathbb{R}$ under the rigid rotation with angle ρ, for any $\rho \in]\frac{p}{q}, \frac{p'}{q'}[$.*

6. APPENDIX

Lemma 18. *Let (X, d) be a metric space and $f : X \to X$ be a continuous map. A positively recurrent point of f is also a positively recurrent point of f^q for all $q \in \mathbb{N}$.*

Proof. If $z \in \mathrm{Rec}^+(f)$, let $O_i = \{z' \in X \mid d(z, z') < \frac{1}{i}\}$ for $i \in \mathbb{N} \setminus \{0\}$. We suppose that $f^{n_k}(z) \to z$ when $k \to +\infty$. Write $n_k = l_k q + p_k$ where $0 \le p_k < q$. If there are infinitely many k such that $p_k = 0$, we are done. Otherwise, there are infinitely many k such that $p_k = p$ where $0 < p < q$. We can suppose that $f^{l_k q + p}(z) \to z$ when $k \to +\infty$ by considering subsequence if necessary. We suppose that $f^{l_{k_1} q + p}(z) \in O_{m_1}$, then there exists O_{m_2} such that $f^{l_{k_1} q+p}(O_{m_2}) \subset O_{m_1}$. Similarly, there exists l_{k_2} and O_{m_3} such that $f^{l_{k_1} q + p}(O_{m_3}) \subset O_{m_2}$. By induction, there is a subsequence $(l_{k_j})_{j\ge1}$ of $(l_k)_{k\ge1}$ and a subsequence $\{O_{m_j}\}_{j\ge1}$ of $\{O_m\}_{m\ge1}$ such that $f^{l_{k_j} q+p}(O_{m_{j+1}}) \subset O_{m_j}$. Consider the subsequence $\{f^{q(p+\sum_{j=(t-1)q}^{tq-1} l_{k_j})}(z)\}_{t\ge1}$, we are done. □

Lemma 19. *Let $f \in \mathrm{Homeo}_*(\mathbb{A})$ and F be a lift of f to $\mathbb{R}^2$. Define the homeomorphism f_q of the annulus $\mathbb{A}_q = \mathbb{R}^2/T^q$ lifted by F and write $\pi_q : \mathbb{R}^2 \to \mathbb{A}_q$ for the covering map. If $z \in \mathrm{Rec}^+(f)$ and $\widetilde{z} \in \pi^{-1}(z)$, then $\pi_q(\widetilde{z}) \in \mathrm{Rec}^+(f_q^q)$.*

Proof. To prove the lemma, we must find two sequences $\{i_n\}_{n\ge1}$ and $\{j_n\}_{n\ge1}$ such that $j_n \to +\infty$ and

$$T^{-qi_n} \circ F^{qj_n}(\widetilde{z}) \to \widetilde{z} \quad (\text{when} \quad n \to +\infty). \tag{6.0.9}$$

Since $z \in \mathrm{Rec}^+(f^q)$, there exist two sequences of integers $\{i_n\}_{n\ge1}$ and $\{j_n\}_{n\ge1}$ such that $j_n \to +\infty$ and $T^{-i_n} \circ F^{qj_n}(\widetilde{z}) \to \widetilde{z}$ when n tends to $+\infty$. Let $\widetilde{O}_i = \{\widetilde{z}' \in \mathbb{R}^2 \mid d(\widetilde{z}, \widetilde{z}') < \frac{1}{i}\}$ for $i \in \mathbb{N} \setminus \{0\}$ where d is the Euclidean metric of $\mathbb{R}^2$. We suppose that $F^{qj_n}(\widetilde{z}) \in T^{i_n}(\widetilde{O}_n)$ for every n by considering subsequence if necessary. Write $i_n = qs_n + t_n$ where $0 \le t_n < q$. If there are infinitely many n such that $t_n = 0$, we are done. Otherwise, there are infinitely

many n such that $t_n = p$ where $0 < p < q$. We can suppose that $F^{qj_n}(\widetilde{z}) \in T^{qs_n+p}(\widetilde{O}_n)$ for every n by further considering subsequence if necessary.

We begin with $F^{qj_{n_1}}(\widetilde{z}) \in T^{qs_{n_1}+p}(\widetilde{O}_{n_1})$, then there exists $\widetilde{O}_{n_2}$ such that $F^{qj_{n_1}}(\widetilde{O}_{n_2}) \subset T^{qs_{n_1}+p}(\widetilde{O}_{n_1})$. Similarly, there exists $\widetilde{O}_{n_3}$ such that $F^{qj_{n_2}}(\widetilde{O}_{n_3}) \subset T^{qs_{n_2}+p}(\widetilde{O}_{n_2})$. By induction, there is a subsequence $\{j_{n_i}\}_{i\geq 1}$ of $\{j_n\}_{n\geq 1}$, a subsequence $\{s_{n_i}\}_{i\geq 1}$ of $\{s_n\}_{n\geq 1}$ and a subsequence $\{\widetilde{O}_{n_i}\}_{i\geq 1}$ of $\{\widetilde{O}_n\}_{n\geq 1}$ such that $F^{qj_{n_i}}(\widetilde{O}_{n_{i+1}}) \subset T^{qs_{n_i}+p}(\widetilde{O}_{n_i})$. Consider the subsequence $\left\{T^{-q(p+\sum_{i=(t-1)q}^{tq-1} s_{n_i})} \circ F^{q(\sum_{i=(t-1)q}^{tq-1} j_{n_i})}(\widetilde{z})\right\}_{t\geq 1}$, we are done. $\square$

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